TEMPERATURE FIELD OF PARALLELEPIPED WITH VOLUME ENERGY SOURCES AND INTERNAL CONVECTION

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UDC 536.23:536.25

Equations are formulated which describe the transfer of heat from a solid core in a parallelepiped to a cooling fluid passing through the parallelepiped. Kantorovich's method of reduction to ordinary differential equations is used to obtain an approximate solution of the equations.

A problem frequently encountered in practice is that of the transfer of heat from a system of spatially distributed objects to a flow of cooling fluid (gas). Such problems crop up, in particular, in the determination of the temperature fields in radio-electronic devices, which have a relatively large number of components mounted with a substantially uniform density. In many cases (in radio-electronic apparatus, for example) the system to be cooled is situated in a container and is itself a more or less uniform generator of energy. The flow of cooling fluid is directed along one of the axes of the parallelepiped. If the components of the system are much smaller than the overall size of the block on which they are mounted, the block can be regarded as approximately uniform and can be described by means of certain effective parameters (thermal conductivity, specific heat capacitance, heat-source density, volume coefficient of internal convection, and so on). Methods are known [1] by means of which various types of distributed systems can be reduced to a uniform block with an effective thermal conductivity and an effective heat-source density. The formulas for the effective thermal conductivity take account of heat transfer by radiation.

The volume coefficient of heat transfer (α_V) is either known directly [2] or can be expressed in terms of the known surface coefficient of heat transfer (α_S)

$$\alpha_V = \alpha_S \frac{S_V}{V}\ .$$

The effective parameters of the equivalent uniform block can thus be determined in many practical cases. Analysis of the temperature field of the distributed system then reduces to solving a set of two equations, the thermal conductivity equation of the block with volume convection taken into account, and the block-coolant heat-transfer equation

$$\frac{\lambda_x}{L_x^2}\frac{\partial^2\vartheta_w}{\partial x^2} + \frac{\lambda_y}{L_y^2}\frac{\partial^2\vartheta_w}{\partial y^2} + \frac{\lambda_z}{L_z^2}\frac{\partial^2\vartheta_w}{\partial z^2} - \alpha_V(\vartheta_w - \vartheta_f) = -q; \tag{1a}$$

$$\vartheta_w = \vartheta_f + \frac{1}{\Omega}\frac{\partial\vartheta_f}{\partial x}; \tag{1b}$$

$$\frac{1}{\Omega} = \frac{c_p\gamma u}{L_x\alpha_V},\quad \vartheta_w = t_w - t_c,\quad \vartheta_f = t_f - t_c. \tag{1c}$$

In the subsequent discussion we shall assume that u = const; q = const.

Equations (1a) and (1b) constitute a closed set of differential equations for ϑ_w and ϑ_f. If the heat transfer from the parallelepiped to the surrounding medium follows Newton's law with the heat-transfer coefficients being identical in pairs at opposite faces, and if the temperature of the cooling fluid at the inlet equals the temperature of the surrounding medium, the boundary conditions of the problem can be

Institute of Exact Mechanics and Optics, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 3, pp. 506-511, September, 1969. Original article submitted May 16, 1968.

written in the form:

$$\left[\frac{\partial \vartheta_w}{\partial \bar{j}} + 2\mathrm{Bi}_j \vartheta_w\right]_{\bar{j}=1} = 0, \quad \left[\frac{\partial \vartheta_w}{\partial \bar{j}} - 2\mathrm{Bi}_j \vartheta_w\right]_{\bar{j}=0} = 0, \tag{2a}$$

$$\mathrm{Bi}_j = \frac{\alpha_j L_j}{2\lambda_j}, \quad j = x, y, z;$$

$$[\vartheta_f]_{\bar{x}=0} = 0. \tag{2b}$$

As far as we are aware no precise solution of the above problem has been published in the literature.

We aim to solve the problem by Kantorovich's method of reduction to ordinary differential equations [3], which is sufficiently accurate for engineering calculations. The error associated with the first approximation of this method is comparable with the error in the determination of the thermophysical parameters λ, α and with the error involved in making the transition to an equivalent uniform block. Consequently, for the present purposes there is no point in trying to find solutions better than the first approximation.

We seek the functions ϑ_w and ϑ_f in the form:

$$\vartheta_w = \sum_{i=1}^{n} f_{xi}\varphi_{yi}\varphi_{zi}; \quad \vartheta_f = \sum_{i=1}^{n} g_{xi}\varphi_{yi}\varphi_{zi}.$$

Here $\varphi_{yi}(\bar{y})$ and $\varphi_{zi}(\bar{z})$ are chosen so as to satisfy the requirements of completeness [3] and boundary conditions (2a); the functions $f_{xi}(\bar{x})$, $g_{xi}(\bar{x})$ are found from the orthogonality conditions

$$\int_0^1\int_0^1 L(\vartheta_w)\varphi_{yi}\varphi_{zi}d\bar{y}\,d\bar{z} = 0, \tag{3}$$

where

$$L(\vartheta_w) = \frac{\lambda_x}{L_x^2}\frac{\partial^2\vartheta_w}{\partial \bar{x}^2} + \frac{\lambda_y}{L_y^2}\frac{\partial^2\vartheta_w}{\partial \bar{y}^2} + \frac{\lambda_z}{L_z^2}\frac{\partial^2\vartheta_w}{\partial \bar{z}^2} - \alpha_V(\vartheta_w - \vartheta_f) + q. \tag{4}$$

Restricting the discussion to the first approximation and dropping in future the indices characterizing the number of the approximation, we can write

$$\vartheta_w = f_x\varphi_y\varphi_z; \quad \vartheta_f = g_x\varphi_y\varphi_z. \tag{5a}$$

If φ_j ($j = y, z$) is given in the form of a square-law parabola, we have from boundary conditions (2a) that

$$\varphi_j = 1 + 2\mathrm{Bi}_j(\bar{j} - \bar{j}^2), \quad \bar{j} = \bar{y}, \bar{z}, \tag{5b}$$

and consequently

$$L(\vartheta_w) = f_x'' M_x\varphi_y\varphi_z - f_x(G_y\varphi_z + G_z\varphi_y) - R_x g_x'\varphi_y\varphi_z + q,$$

$$M_x = \frac{\lambda_x}{L_x^2}, \quad R_x = \frac{\alpha_V}{\Omega}, \quad G_j = \frac{4\mathrm{Bi}_j\lambda_j}{L_j^2}, \quad j = y, z.$$

Utilizing (3) and writing

$$I_{1j} = \int_0^1 \varphi_j^2 d\bar{j} = 1 + \frac{2}{3}\mathrm{Bi}_j + \frac{2}{15}\mathrm{Bi}_j^2; \tag{6a}$$

$$I_{2j} = \int_0^1 \varphi_j d\bar{j} = 1 + \frac{1}{3}\mathrm{Bi}_j, \tag{6b}$$

we obtain

$$f_x'' - \frac{R_x}{M_x}g_x' - \frac{1}{M_x}\left(G_y\frac{I_{2y}}{I_{1y}} + G_z\frac{I_{2z}}{I_{1z}}\right)f_x = -\frac{q}{M_x}\frac{I_{2y}I_{2z}}{I_{1y}I_{1z}}. \tag{7a}$$

Insertion of (5a) into (1b) gives

$$f_x = g_x + \frac{g_x'}{\Omega}. \tag{7b}$$

Elimination of the function f_x between Eqs. (7a) and (7b) gives the following final differential equation for g_x:

$$g_x''' + g_x'' C_2 + g_x' C_1 - g_x C_0 = -D; \tag{8a}$$

$$\left.\begin{aligned} C_2 = \Omega;\ C_1 &= \frac{1}{M_x}\left(\alpha_V + \frac{G_y I_{2y}}{I_{1y}} + \frac{G_z I_{2z}}{I_{1z}}\right), \\ C_0 = \frac{\Omega}{M_x}\left(\frac{G_y I_{2y}}{I_{1y}} + \frac{G_z I_{2z}}{I_{1z}}\right);\ & D = q\Omega \frac{I_{2y} I_{2z}}{I_{1y} I_{1z}}. \end{aligned}\right\} \tag{8b}$$

The general integral of the linear differential equation (8a) has the form

$$g_x = M_1 \exp(k_1 \bar{x}) + M_2 \exp(k_2 \bar{x}) + M_3 \exp(k_3 \bar{x}) + \frac{D}{C_0}, \tag{9}$$

where M_1, M_2, M_3 are arbitrary constants, determined from the boundary conditions; k_1, k_2, k_3 are the roots of the characteristic equation

$$k^3 + C_2 k^2 - C_1 k - C_0 = 0. \tag{10a}$$

We introduce the new variable

$$\xi = k + \frac{C_2}{3}, \tag{10b}$$

and rewrite (10a) in the form [4]

$$\xi^3 + 3\beta\xi + 2\varepsilon = 0; \tag{10c}$$

$$\varepsilon = \frac{C_2^3}{27} + \frac{C_2 C_1}{6} - \frac{C_0}{2};\quad \beta = -\frac{C_1}{3} - \frac{C_2^2}{9}. \tag{10d}$$

The formulas for k will depend on the signs of ε, β, and of the discriminant $d = \beta^3 + \varepsilon^2$. Clearly, from (10d) we have always that

$$\beta \leqslant 0.$$

Determination of the signs of ε and d requires a more complex analysis. On substitution of expressions (8b) into (10d) we obtain after some manipulation

$$\varepsilon = \frac{\Omega^3}{27} + \frac{\Omega}{3}(\mathrm{Bi}_V - A); \tag{11a}$$

$$\begin{aligned} d &= \frac{C_0^2}{4} - \left(\frac{C_1^3}{27} + \frac{C_1^2 C_2^2}{108} + \frac{C_2^3 C_0}{27} + \frac{C_0 C_1 C_2}{6}\right) = \\ &= \frac{\Omega^2 A^2}{4} - \left[\frac{(\mathrm{Bi}_V + A)^3}{27} + \frac{(\mathrm{Bi}_V + A)^2 \Omega^2}{108} + \frac{\Omega^2 A}{27} + \frac{\Omega^2 A(\mathrm{Bi}_V + A)}{6}\right]; \end{aligned} \tag{11b}$$

$$A = 4\left(\frac{\lambda_y}{\lambda_x}\frac{L_x^2}{L_y^2}\mathrm{Bi}_y\frac{I_{2y}}{I_{1y}} + \frac{\lambda_z}{\lambda_x}\frac{L_x^2}{L_z^2}\mathrm{Bi}_z\frac{I_{2z}}{I_{1z}}\right); \tag{11c}$$

$$\mathrm{Bi}_V = \frac{\alpha_V L_x^2}{\lambda_x}. \tag{11d}$$

We note that A is independent of Bi_V and Ω and that

$$\lim_{u\to\infty}\Omega = 0,\quad \lim_{\alpha_V\to 0}\Omega = 0,\quad \lim_{u\to 0}\Omega = \infty,\quad \lim_{\alpha_V\to\infty}\Omega = \infty,$$

$$\lim_{\substack{\mathrm{Bi}_y\to 0\\ \mathrm{Bi}_z\to 0}} A = 0,\quad \lim_{\substack{\mathrm{Bi}_y\to\infty\\ \mathrm{Bi}_z\to\infty}} A = 10\left(\frac{\lambda_y}{L_y^2}\frac{L_x^2}{\lambda_x} + \frac{\lambda_z L_x^2}{L_z^2\lambda_x}\right),$$

since

$$\lim_{\mathrm{Bi}_j\to\infty}\frac{\mathrm{Bi}_j\left(1 + \frac{1}{3}\mathrm{Bi}_j\right)}{1 + \frac{2}{3}\mathrm{Bi}_j + \frac{2}{15}\mathrm{Bi}_j^2} = \frac{5}{2}.$$

Consequently, for the limiting cases $\alpha_V \to 0, \infty$ or $u \to 0, \infty$ we have that

$$d \leqslant 0, \ \varepsilon \geqslant 0. \tag{12}$$

Determination of the signs of d and ε is troublesome in the case of intermediate values of α_V, u.

For convenience the rest of the discussion will be carried out with a specific class of objects in mind. For example, for radio-electronic apparatus the parameter values are such that

$$\mathrm{Bi}_V > 5A$$

so that inequalities (12) are satisfied. We then have [4]:

$$\left.\begin{aligned} &k_i = \xi_i - \frac{\Omega}{3} \ (i = 1, 2, 3); \ r = \sqrt{|\beta|}; \ \varphi = \arccos\frac{\varepsilon}{r^3}; \\ &\xi_1 = -2r\cos\frac{\varphi}{3}; \ \xi_2 = 2r\cos\left(\frac{\pi}{3} - \frac{\varphi}{3}\right); \ \xi_3 = 2r\cos\left(\frac{\pi}{3} + \frac{\varphi}{3}\right). \end{aligned}\right\} \tag{13}$$

For simplicity we introduce the notation:

$$h \equiv g\frac{C_0}{D}; \ P_{1,2,3} = M_{1,2,3}\frac{C_0}{D}; \ F = f_x\frac{C_0}{D}, \tag{14}$$

when (9) and (7b) can be rewritten:

$$h = P_1\exp(k_1\bar{x}) + P_2\exp(k_2\bar{x}) + P_3\exp(k_3\bar{x}) + 1,$$

$$F = P_1\exp(k_1\bar{x})\left(1 + \frac{k_1}{\Omega}\right) + P_2\exp(k_2\bar{x})\left(1 + \frac{k_2}{\Omega}\right) + P_3\exp(k_3\bar{x})\left(1 + \frac{k_3}{\Omega}\right) + 1.$$

To determine the constants $P_{1,2,3}$ we make use of the boundary conditions. From (2b) in conjunction with (5a) and (14) we obtain: $h(\bar{x} = 0) = 0$, from which

$$P_1 + P_2 + P_3 = -1. \tag{15a}$$

From conditions (2a) in conjunction with (5a), (14), and (9b) we also have

$$\begin{aligned} P_1\left[\exp(k_1)\left(1 + \frac{k_1}{\Omega}\right)(k_1 + 2\mathrm{Bi}_x)\right] + P_2\left[\exp(k_2)\left(1 + \frac{k_2}{\Omega}\right)(k_2 + 2\mathrm{Bi}_x)\right] \\ + P_3\left[\exp(k_3)\left(1 + \frac{k_3}{\Omega}\right)(k_3 + 2\mathrm{Bi}_x)\right] = -2\mathrm{Bi}_x; \end{aligned} \tag{15b}$$

$$P_1\left[\left(1 + \frac{k_1}{\Omega}\right)(k_1 - 2\mathrm{Bi}_x)\right] + P_2\left[\left(1 + \frac{k_2}{\Omega}\right)(k_2 - 2\mathrm{Bi}_x)\right] + P_3\left[\left(1 + \frac{k_3}{\Omega}\right)(k_3 - 2\mathrm{Bi}_x)\right] = 2\mathrm{Bi}_x. \tag{15c}$$

Equations (15a, b, c) determine the constants P_1, P_2, P_3.

In a number of cases it is of interest to determine the mean temperature

$$\bar{F} = \int_0^1 F d\bar{x} = \frac{P_1}{k_1}\left(1 + \frac{k_1}{\Omega}\right)[\exp(k_1) - 1] + \frac{P_2}{k_2}\left(1 + \frac{k_2}{\Omega}\right)[\exp(k_2) - 1] + \frac{P_3}{k_3}\left(1 + \frac{k_3}{\Omega}\right)[\exp(k_3) - 1] + 1; \tag{16a}$$

$$\bar{h} = \int_0^1 h d\bar{x} = \frac{P_1}{k_1}[\exp(k_1) - 1] + \frac{P_2}{k_2}[\exp(k_2) - 1] + \frac{P_3}{k_3}[\exp(k_3) - 1]. \tag{16b}$$

If the cooling fluid moves through natural convection, the ratio of the difference between the mean block and mean fluid temperatures to the mean block temperature is an important parameter upon which the total hydraulic pressure head of the flow depends. This ratio is given by:

$$\varkappa \equiv \frac{\bar{\vartheta}_w - \bar{\vartheta}_f}{\bar{\vartheta}_w} = \frac{\int_0^1\int_0^1\int_0^1 \vartheta_w d\bar{x}\, d\bar{y}\, d\bar{z} - \int_0^1\int_0^1\int_0^1 \vartheta_f d\bar{x}\, d\bar{y}\, d\bar{z}}{\int_0^1\int_0^1\int_0^1 \vartheta_w d\bar{x}\, d\bar{y}\, d\bar{z}} = \frac{i}{\Omega} \frac{P_1 \exp(k_1) + P_2 \exp(k_2) + P_3 \exp(k_3)}{\bar{F}}. \tag{17}$$

Finally,

$$\vartheta_w = \frac{D}{C_0} \varphi_y \varphi_z F, \quad \vartheta_f = \frac{D}{C_0} \varphi_y \varphi_z h; \tag{18a}$$

$$\bar{\vartheta}_w = \frac{D}{C_0} \int_0^1 \varphi_y d\bar{y} \int_0^1 \varphi_z d\bar{z}\, \bar{F} = \frac{D}{C_0} I_{2y} I_{2z} \bar{F}; \tag{18b}$$

$$\bar{\vartheta}_f = \frac{D}{C_0} I_{2y} I_{2z} \bar{h}, \tag{18c}$$

where D and C_0 are given by formulas (8b); φ_y, φ_z by formulas (5b); I_{2y}, I_{2z} by (6b); h, F by (9a, b); $\bar{h}$, $\bar{F}$ by (16a, b); the exponents $k_{1,2,3}$ by (13), (10d), (8b); the constants $P_{1,2,3}$ by (15a, b, c); by (1c); Bi_j by (2a); and Bi_V by (11d).

The above formulas were used to construct a program for temperature field calculations on the Minsk-2 digital computer. Calculations were carried out for a radio-electronic device of cassette construction. The satisfactory agreement which was obtained between computation and experiment enables the above method to be recommended for the study of distributed systems with internal convection. The limits of applicability of the formulas are set by the assumptions made in their derivation.

NOTATION

α_V, α_S	are the volume and surface heat-transfer coefficients;
λ_f	is the coefficient of thermal conductivity of cooling fluid;
S_V	is the total surface area of elements of block;
V	is the volume of elements;
L_x, L_y, L_z	are the lengths of sides of parallelepiped in directions of x, y, z axes;
q	is the density of volumetric energy sources;
$q\lambda_x$, λ_y, λ_z	are the coefficients of thermal conductivity of anisotropic body in x, y, z directions;
Bi	is the Biot number;
$t_w(x, y, z)$	is the temperature of block at point (x, y, z);
t_c	is the temperature of medium surrounding block;
$t_f(x, y, z)$	is the temperature of cooling fluid at point (x, y, z);
u	is the mean velocity of fluid flow in x direction;
$\bar{x} = x/L_x$;	
$\bar{y} = y/L_y$;	
$\bar{z} = z/L_z$	are the relative coordinates;
α	is the coefficient of heat transfer to medium outside parallelepiped.

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